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SHARP UPPER BOUNDS FOR A SINGULAR PERTURBATION PROBLEM RELATED TO MICROMAGNETICS

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1. introduction

In this paper we study the following energy-functional, related to micromagnetics:

$$E_\varepsilon(u) := \int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} |H_u|^2. \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary, u is a unit-valued vector-field (corresponding to the magnetization) in $H^1(\Omega, S^1)$ and H_u, the demagnetizing field created by u, is given by

$$\begin{cases} div\,(\tilde u + H_u) = 0 & \text{in } \mathbb{R}^2 \\ curl\, H_u = 0 & \text{in } \mathbb{R}^2\,, \end{cases} \tag{1.2}$$

where $\tilde u$ is the extension of u by 0 in $\mathbb{R}^2 \setminus \Omega$. For the physical models related to E_ε, we refer to [18] and all the references therein.

We can rewrite (1.1) in the following form. Denoting by $\Delta^{-1}\tilde u$ the Newtonian potential of $\tilde u$, we observe that the vector-field $\bar H_u := -\nabla\big(div(\Delta^{-1}\tilde u)\big)$ belongs to $L^2(\mathbb{R}^2, \mathbb{R}^2)$. Moreover,

$$\begin{cases} div\, \bar H_u = -div\,\tilde u & \text{in } \mathbb{R}^2 \\ curl\, \bar H_u = 0 & \text{in } \mathbb{R}^2\,. \end{cases}$$

So $H_u = \bar H_u$ and we obtain

$$E_\varepsilon(u) = \int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} \big|\nabla\big(div(\Delta^{-1}\tilde u)\big)\big|^2. \tag{1.3}$$

In [19] T.Rivière and S.Serfaty proved the following theorem, giving compactness and a lower bound for the energies E_ε.

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Theorem 1.1. *Let Ω be a bounded simply connected domain in $\mathbb{R}^2$. Let $\varepsilon_n \to 0$ and $u_n \in H^1(\Omega, S^1)$ with a lifting $\varphi_n \in H^1(\Omega, \mathbb{R})$ i.e. $u_n = e^{i\varphi_n}$ a.e., and such that*

$$E_{\varepsilon_n}(u_n) \le C \tag{1.4}$$

$$\|\varphi_n\|_{L^\infty(\Omega)} \le N\,. \tag{1.5}$$

Then, up to extraction of a subsequence, there exists u and φ in $\cap_{q<\infty} L^q(\Omega)$ such that

$$\varphi_n \to \varphi \ \text{in}\ \cap_{q<\infty} L^q(\Omega)$$

$$u_n \to u \ \text{in}\ \cap_{q<\infty} L^q(\Omega)\,.$$

Moreover, if we consider

$$\begin{cases} T^t\varphi(x) := \inf\big(\varphi(x), t\big) \\ T^t u(x) := e^{\,i\,T^t\varphi(x)}\,, \end{cases}$$

then $div_x T^t u$ is a bounded Radon measure on $\Omega \times \mathbb{R}$, with $t \mapsto div_x T^t u$ continuous from $\mathbb{R}$ to $\mathcal{D}'(\Omega)$. In addition

$$2\int_{\mathbb{R}}\int_{\Omega} \big|div_x\, T^t u\big|\, dx\, dt \le \varliminf_{n\to\infty} \int_{\Omega} 2|\nabla\varphi_n \cdot H_{u_n}| \le \varliminf_{n\to\infty} E_{\varepsilon_n}(u_n) < \infty\,.$$

The main contribution of this paper is to establish the upper bound for E_ε in the case where u and its lifting φ belong to BV. First of all we want to observe that if $u_n \to u$ in L^q, where $|u_n| = 1$, $u \in BV$ and $E_{\varepsilon_n}(u_n) \le C$, then clearly $div\ \tilde{u} = 0$ as a distribution, i.e.

$$\begin{cases} |u| = 1 \quad \text{a.e. in } \Omega\,, \\ div\ u = 0 \quad \text{in } \Omega\,, \\ u \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega\,. \end{cases} \tag{1.6}$$

The main result of this paper is the following theorem.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary. Consider $u \in BV(\Omega, S^1)$, satisfying $div\, u = 0$ in Ω and $u \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ and assume there exist $\varphi \in BV(\Omega, \mathbb{R})$, such that $u = e^{i\varphi}$ a e. in Ω. Then there exists a family of functions $\{v_\varepsilon\} \subset C^2(\mathbb{R}^N, \mathbb{R})$ satisfying*

$$\lim_{\varepsilon\to 0^+} v_\varepsilon(x) = \varphi(x) \quad \text{in} \quad L^1(\Omega, \mathbb{R})$$

and

$$\lim_{\varepsilon\to 0} E_\varepsilon(e^{iv_\varepsilon}) = 2\int_{\mathbb{R}}\int_{\Omega} \big|div_x\, T^t u\big|\, dx\, dt\,.$$

Moreover, if $\varphi \in BV(\Omega,\mathbb{R})\cap L^\infty$, then we have

$$\lim_{\varepsilon\to 0^+} v_\varepsilon(x) = \varphi(x) \quad in \quad L^p(\Omega,\mathbb{R}) \quad \forall p\in[1,\infty)\,.$$

In order to construct $\{v_\varepsilon\}$ we take the convolution of φ with a varying smoothing kernel, i.e. we set $v_\varepsilon(x) := \varepsilon^{-2}\int_{\mathbb{R}^2}\eta\big(\frac{y-x}{\varepsilon},x\big)\varphi(y)dy$, and we optimize the choice of the kernel. A similar approach was used in [16] and [17], but a new ingredient is required here, since the non-local term $\int_{\mathbb{R}^2}|H_u|^2$ gives more difficulties.

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2. PRELIMINARIES

Throughout this section we assume that Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary. We begin by introducing some notation. For every $\boldsymbol{\nu}\in S^1$ (the unit sphere in $\mathbb{R}^2$) and $R>0$ we denote

$$B_R^+(x,\boldsymbol{\nu}) = \{y\in\mathbb{R}^2 \,:\, |y-x|<R,\ (y-x)\cdot\boldsymbol{\nu}>0\}\,, \tag{2.1}$$

$$B_R^-(x,\boldsymbol{\nu}) = \{y\in\mathbb{R}^2 : |y-x|<R,\ (y-x)\cdot\boldsymbol{\nu}<0\}\,, \tag{2.2}$$

$$H_+(x,\boldsymbol{\nu}) = \{y\in\mathbb{R}^2 : (y-x)\cdot\boldsymbol{\nu}>0\}\,, \tag{2.3}$$

$$H_-(x,\boldsymbol{\nu}) = \{y\in\mathbb{R}^2 : (y-x)\cdot\boldsymbol{\nu}<0\} \tag{2.4}$$

and

$$H^0_{\boldsymbol{\nu}} = \{y\in\mathbb{R}^2 \,:\, y\cdot\boldsymbol{\nu}=0\}\,. \tag{2.5}$$

Definition 2.1. Consider a function $f\in BV(\Omega,\mathbb{R}^m)$ and a point $x\in\Omega$.
i) We say that x is a point of *approximate continuity* of f if there exists $z\in\mathbb{R}^m$ such that

$$\lim_{\rho\to 0^+}\frac{\int_{B_\rho(x)}|f(y)-z|\,dy}{\mathcal{L}^2\big(B_\rho(x)\big)} = 0\,.$$

In this case z is called an *approximate limit* of f at x and we denote z by $\tilde f(x)$. The set of points of approximate continuity of f is denoted by G_f.

ii) We say that x is an *approximate jump point* of f if there exist $a, b \in \mathbb{R}^m$ and $\boldsymbol{\nu} \in S^{N-1}$ such that $a \neq b$ and

$$\text{(2.6)} \qquad \lim_{\rho\to 0^+} \frac{\int_{B_\rho^+(x,\boldsymbol{\nu})} |f(y)-a|\,dy}{\mathcal{L}^2\big(B_\rho(x)\big)} = 0, \qquad \lim_{\rho\to 0^+} \frac{\int_{B_\rho^-(x,\boldsymbol{\nu})} |f(y)-b|\,dy}{\mathcal{L}^2\big(B_\rho(x)\big)} = 0.$$

The triple $(a, b, \boldsymbol{\nu})$, uniquely determined by (2.6) up to a permutation of (a, b) and a change of sign of $\boldsymbol{\nu}$, is denoted by $(f^+(x), f^-(x), \boldsymbol{\nu}_f(x))$. We shall call $\boldsymbol{\nu}_f(x)$ the *approximate jump vector* and we shall sometimes write simply $\boldsymbol{\nu}(x)$ if the reference to the function f is clear. The set of approximate jump points is denoted by J_f. A choice of $\boldsymbol{\nu}(x)$ for every $x \in J_f$ (which is unique up to sign) determines an orientation of J_f. At a point of approximate continuity x, we shall use the convention $f^+(x) = f^-(x) = \tilde{f}(x)$.

We refer to [2] for the results on BV-functions that we shall use in the sequel.

Consider a function $\Phi = (\varphi_1, \varphi_2, \dots, \varphi_d) \in BV(\Omega, \mathbb{R}^d)$. By [2, Proposition 3.21] we may extend Φ to a function $\bar{\Phi} \in BV(\mathbb{R}^2, \mathbb{R}^d)$, such that $\bar{\Phi} = \Phi$ a.e. in Ω, $\operatorname{supp}\bar{\Phi}$ is compact and $\|D\bar{\Phi}\|(\partial\Omega) = 0$. From the proof of Proposition 3.21 in [2] it follows that if $\Phi \in BV(\Omega, \mathbb{R}^d) \cap L^\infty$ then its extension $\bar{\Phi}$ is also in $BV(\mathbb{R}^2, \mathbb{R}^d) \cap L^\infty$. Consider also a matrix valued function $\Xi \in C_c^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^{l\times d})$ For every $\varepsilon > 0$ define a function $\Psi_\varepsilon(x) : \mathbb{R}^2 \to \mathbb{R}^l$ by

$$\text{(2.7)} \quad \Psi_\varepsilon(x) := \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \Xi\Big(\frac{y-x}{\varepsilon}, x\Big)\cdot\bar{\Phi}(y)dy = \int_{\mathbb{R}^2} \Xi(z,x)\cdot\bar{\Phi}(x+\varepsilon z)\,dz, \quad \forall x \in \mathbb{R}^2\,.$$

Due to [17] (Proposition 3.2), we have the following statement.

Proposition 2.1. *Let $W \in C^1(\mathbb{R}^l \times \mathbb{R}^q, \mathbb{R})$ satisfying*

$$\text{(2.8)} \qquad \nabla_a W(a,b) = 0 \quad \textit{whenever } W(a,b) = 0\,.$$

Consider $\Phi \in BV(\Omega, \mathbb{R}^d) \cap L^\infty$ and $u \in BV(\Omega, \mathbb{R}^q) \cap L^\infty$ satisfying

$$W\bigg(\Big\{\int_{\mathbb{R}^2} \Xi(z,x)\,dz\Big\}\cdot\Phi(x),\ u(x)\bigg) = 0 \quad \textit{for a.e. } x \in \Omega\,,$$

where $\Xi \in C_c^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^{l\times d})$, as above. Let Ψ_ε be as in (2.7). Then,

$$\text{(2.9)} \quad \lim_{\varepsilon\to 0}\int_\Omega \frac{1}{\varepsilon} W\big(\Phi_\varepsilon(x), u(x)\big)dx =$$
$$\int_{J_\Phi}\Big\{\int_{-\infty}^0 W\big(\Gamma(t,x), u^+(x)\big)dt + \int_0^{+\infty} W\big(\Gamma(t,x), u^-(x)\big)dt\Big\}d\mathcal{H}^1(x)\,,$$

where

$$\Gamma(t,x) = \left(\int_{-\infty}^{t} P(s,x)\,ds\right)\cdot\Phi^-(x) + \left(\int_{t}^{+\infty} P(s,x)\,ds\right)\cdot\Phi^+(x), \tag{2.10}$$

with

$$P(t,x) = \int_{H^0_{\boldsymbol{\nu}(x)}} \Xi(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\,, \tag{2.11}$$

$\boldsymbol{\nu}(x)$ *is the jump vector of* Φ *and it is assumed that the orientation of* J_u *coincides with the orientation of* J_Φ $\mathcal{H}^1$ *a.e. on* $J_u \cap J_\Phi$.

Definition 2.2. Given $f \in L^\infty(\mathbb{R}^2,\mathbb{R}^k)$ with compact support, we define its Newtonian potential

$$(\Delta^{-1}f)(x) := \int_{\mathbb{R}^2} \frac{1}{2\pi}\ln|x-y|\,f(y)\,dy\,.$$

Than it is well known that

$$\int_{\mathbb{R}^2} \left|\nabla^2(\Delta^{-1}f)(x)\right|^2 dx = \int_{\mathbb{R}^2} |f(x)|^2\,dx\,. \tag{2.12}$$

Definition 2.3. Let $\mathcal{V}$ be the class of all functions $\eta \in C^2_c(\mathbb{R}^2\times\mathbb{R}^2,\mathbb{R})$ such that

$$\int_{\mathbb{R}^2} \eta(z,x)\,dz = 1 \quad \forall x\in\Omega\,. \tag{2.13}$$

Let $\mathcal{U}$ be the class of all functions $l(z,x) \in C^2_c(\mathbb{R}^2\times\Omega,\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} l(z,x)\,dz = 0 \quad \forall x\in\mathbb{R}^2\,. \tag{2.14}$$

In [17] (Lemma 5.1), we proved the following statement.

Lemma 2.1. *Let* μ *be positive finite Borel measure on* Ω *and* $\boldsymbol{\nu}_0(x) : \Omega \to \mathbb{R}^2$ *a Borel measurable function with* $|\boldsymbol{\nu}_0| = 1$. *Let* $\mathcal{W}_1$ *denote the set of functions* $p(t,x) : \mathbb{R}\times\Omega \to \mathbb{R}$ *satisfying the following conditions:*

i) p *is Borel measurable and bounded,*

ii) there exists $M > 0$ *such that* $p(t,x) = 0$ *for* $|t| > M$ *and any* $x\in\Omega$,

iii) $\int_{\mathbb{R}} p(t,x)\,dt = 1$, $\forall x\in\Omega$.

Then for every $p(t,x)\in\mathcal{W}_1$, *there exists a sequence of functions* $\{\eta_n\}\subset\mathcal{V}$ *(see Definition 2.3), such that the sequence of functions* $\{p_n(t,x)\}$ *defined on* $\mathbb{R}\times\Omega$ *by*

$$p_n(t,x) = \int_{H^0_{\boldsymbol{\nu}_0(x)}} \eta_n(t\boldsymbol{\nu}_0(x)+y,x)d\mathcal{H}^1(y),$$

has the following properties:

i) there exists C_0 *such that* $\|p_n\|_{L^\infty} \le C_0$ *for every* n, *ii) there exist $M > 0$ such that $p_n(t,x) = 0$ for $|t| > M$ and every $x \in \Omega$, for all n.*

iii) $\lim_{n\to\infty} \int_\Omega \int_{\mathbb{R}} |p_n(t,x) - p(t,x)| \, dt \, d\mu(x) = 0.$

With the same method it is not difficult to prove

Lemma 2.2. *Let μ be positive finite Borel measure on Ω and $\boldsymbol{\nu}_0(x) : \Omega \to \mathbb{R}^2$ a Borel measurable function with $|\boldsymbol{\nu}_0| = 1$. Let $\mathcal{W}_0$ denote the set of functions $q(t,x) : \mathbb{R} \times \Omega \to \mathbb{R}^2$ satisfying the following conditions:*

i) q is Borel measurable and bounded,

ii) there exists $M > 0$ such that $q(t,x) = 0$ for $|t| > M$ and every $x \in \Omega$.

iii) $\int_{\mathbb{R}} q(t,x) \, dt = 0$, $\forall x \in \Omega$.

Then for every $q(t,x) \in \mathcal{W}_0$, there exists a sequence of functions $\{l_n\} \subset \mathcal{U}$ (see Definition 2.3), such that the sequence of functions $\{q_n(t,x)\}$ defined on $\mathbb{R} \times \Omega$ by

$$q_n(t,x) = \int_{H^0_{\boldsymbol{\nu}_0(x)}} l_n(t\boldsymbol{\nu}_0(x) + y, x) d\mathcal{H}^1(y),$$

has the following properties:

i) there exists C_0 such that $\|q_n\|_{L^\infty} \le C_0$ for every n,

ii) there exist $M > 0$ such that $q_n(t,x) = 0$ for $|t| > M$ and every $x \in \Omega$, for all n,

iii) $\lim_{n\to\infty} \int_\Omega \int_{\mathbb{R}} |q_n(t,x) - q(t,x)| \, dt \, d\mu(x) = 0.$

3. First estimates

Throughout this section we assume that Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary.

Let $l \in \mathcal{U}$ (see Definition 2.3). Consider $r(z,x) := \Delta_z^{-1} l(z,x)$. Then $r \in C^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2)$ with $\operatorname{supp} r \subset \mathbb{R}^2 \times K$, where $K \subset\subset \Omega$. Moreover, since $\int_{\mathbb{R}^2} l(z,x) dz = 0$, for every $k = 0, 1, 2 \ldots$ we have the estimates

(3.1)
$$|\nabla_x^k r(z,x)| \le \frac{C_k}{|z|+1}, \quad \left|\nabla_x^k \big(\nabla_z r(z,x)\big)\right| \le \frac{C_k}{|z|^2+1}, \quad \left|\nabla_x^k \big(\nabla_z^2 r(z,x)\big)\right| \le \frac{C_k}{|z|^3+1},$$

where $C_k > 0$ does not depend on z and x.

Lemma 3.1. *Let $\varphi \in BV(\Omega, \mathbb{R}) \cap L^\infty$ and $l \in \mathcal{U}$ (see Definition 2.3). For every $\varepsilon > 0$ consider the function $\varphi_\varepsilon \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ by*

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} l\Big(\frac{y-x}{\varepsilon}, x\Big) \bar{\varphi}(y) \, dy = \int_{\mathbb{R}^2} l(z,x) \bar{\varphi}(x + \varepsilon z) \, dz \,, \tag{3.2}$$

where $\bar\varphi$ is some bounded BV extension of φ to $\mathbb{R}^2$ with compact support. Next consider $r(z,x) := \Delta_z^{-1} l(z,x)$ and set

$$\xi_\varepsilon(x) := \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \nabla_1\big(div_1\, r\big)\Big(\frac{y-x}{\varepsilon}, x\Big)\bar\varphi(y)\,dy = \int_{\mathbb{R}^2} \nabla_z\big(div_z\, r(z,x)\big)\bar\varphi(x+\varepsilon z)\,dz\,, \tag{3.3}$$

where $\nabla_1(div_1\, r)$ is the gradient of divergence of $r(z,x)$ in its first variable, namely z. Then,

$$\int_{\mathbb{R}^2} \frac{1}{\varepsilon}\Big|\nabla\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx = o_\varepsilon(1) + \int_\Omega \frac{1}{\varepsilon}\,\varphi_\varepsilon(x)\cdot\xi_\varepsilon(x)\,dx\,. \tag{3.4}$$

Proof. Since $l(z,x) = 0$ if $x \notin K$, where K is some compact subset of Ω, we have, in particular, $\varphi_\varepsilon(x) = 0$ for every $x \in \mathbb{R}^2 \setminus \Omega$. Then, integrating by part two times, we conclude

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{1}{\varepsilon}\Big|\nabla\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx = -\int_{\mathbb{R}^2}\frac{1}{\varepsilon}\,\Delta\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\cdot\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\,dx = \\ -\int_{\mathbb{R}^2}\frac{1}{\varepsilon}\,div\,\varphi_\varepsilon(x)\cdot\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\,dx = \int_\Omega \frac{1}{\varepsilon}\,\varphi_\varepsilon(x)\cdot\nabla\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\,dx\,.\end{aligned} \tag{3.5}$$

Next consider the function $\zeta_\varepsilon \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$\zeta_\varepsilon(x) := \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} r\Big(\frac{y-x}{\varepsilon}, x\Big)\bar\varphi(y)\,dy = \int_{\mathbb{R}^2} r(z,x)\bar\varphi(x+\varepsilon z)\,dz\,. \tag{3.6}$$

We will prove now that

$$\Big|\varepsilon^2\nabla^2\zeta_\varepsilon(x) - \int_{\mathbb{R}^2}\nabla_z^2 r(z,x)\bar\varphi(x+\varepsilon z)\,dz\Big| \leq C\varepsilon^{2/3} \quad \forall x \in \Omega\,. \tag{3.7}$$

We shall denote by $\nabla_1 l$ and $\nabla_2 l$ the gradient of $l(z,x)$ w.r.t. the variables z and x respectively. We have,

$$\begin{aligned}\varepsilon^2\nabla^2\zeta_\varepsilon(x) - \int_{\mathbb{R}^2}\nabla_z^2 r(z,x)\bar\varphi(x+\varepsilon z)\,dz = \\ \int_{\mathbb{R}^2}\nabla_x^2 r\Big(\frac{y-x}{\varepsilon}, x\Big)\bar\varphi(y)\,dy - \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\nabla_1^2 r\Big(\frac{y-x}{\varepsilon}, x\Big)\bar\varphi(y)\,dy = \\ -\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\Big\{\nabla_1\nabla_2 r\Big(\frac{y-x}{\varepsilon}, x\Big) + \nabla_2\nabla_1 r\Big(\frac{y-x}{\varepsilon}, x\Big)\Big\}\bar\varphi(y)\,dy + \int_{\mathbb{R}^2}\nabla_2^2 r\Big(\frac{y-x}{\varepsilon}, x\Big)\bar\varphi(y)\,dy\,.\end{aligned} \tag{3.8}$$

Therefore, by the Hölder inequality and the estimates in (3.1), we obtain

$$\left|\varepsilon^2\nabla^2\zeta_\varepsilon(x)-\int_{\mathbb{R}^2}\nabla_z^2 r(z,x)\bar{\varphi}(x+\varepsilon z)\,dz\right|\le$$

$$\varepsilon^{2/3}\Big(\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\Big|\nabla_1\nabla_2 r\Big(\frac{y-x}{\varepsilon},x\Big)+\nabla_2\nabla_1 r\Big(\frac{y-x}{\varepsilon},x\Big)\Big|^{6/5}dy\Big)^{5/6}\Big(\int_{\mathbb{R}^2}|\bar{\varphi}(y)|^6dy\Big)^{1/6}$$

$$+\,\varepsilon^{2/3}\Big(\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\Big|\nabla_2^2 r\Big(\frac{y-x}{\varepsilon},x\Big)\Big|^3dy\Big)^{1/3}\Big(\int_{\mathbb{R}^2}|\bar{\varphi}(y)|^{3/2}dy\Big)^{2/3}=$$

$$\varepsilon^{2/3}\Big(\int_{\mathbb{R}^2}\Big|\nabla_1\nabla_2 r\big(z,x\big)+\nabla_2\nabla_1 r\big(z,x\big)\Big|^{6/5}dz\Big)^{5/6}\Big(\int_{\mathbb{R}^2}|\bar{\varphi}(y)|^6dy\Big)^{1/6}+$$

$$\varepsilon^{2/3}\Big(\int_{\mathbb{R}^2}\Big|\nabla_2^2 r\big(z,x\big)\Big|^3dz\Big)^{1/3}\Big(\int_{\mathbb{R}^2}|\bar{\varphi}(y)|^{3/2}dy\Big)^{2/3}\le C\varepsilon^{2/3}$$

which gives (3.7). In particular,

$$(3.9)\quad \Big|\varepsilon^2\Delta\zeta_\varepsilon(x)\,-\,\varphi_\varepsilon(x)\Big|\;=\;\Big|\varepsilon^2\Delta\zeta_\varepsilon(x)\,-\,\int_{\mathbb{R}^2}\Delta_z r(z,x)\bar{\varphi}(x\,+\,\varepsilon z)\,dz\Big|\;\le\;C_0\varepsilon^{2/3}\,.$$

Next by (3.5),

$$(3.10)\quad \int_{\mathbb{R}^2}\frac{1}{\varepsilon}\Big|\nabla\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2dx=\int_\Omega\frac{1}{\varepsilon}\,\varphi_\varepsilon(x)\cdot\nabla\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\,dx$$

$$=\int_\Omega\frac{1}{\varepsilon}\,\varphi_\varepsilon(x)\cdot\nabla\Big(div\big(\varepsilon^2\zeta_\varepsilon(x)\big)\Big)(x)\,dx-\int_\Omega\frac{1}{\varepsilon}\,\varphi_\varepsilon(x)\cdot\nabla\Big(div\Big(\Delta^{-1}\big(\varepsilon^2\Delta\zeta_\varepsilon-\varphi_\varepsilon\big)\Big)\Big)(x)\,dx\,.$$

The last integral can be estimated by

$$\Big|\int_\Omega\frac{1}{\varepsilon}\,\varphi_\varepsilon(x)\cdot\nabla\Big(div\Big(\Delta^{-1}\big(\varepsilon^2\Delta\zeta_\varepsilon-\varphi_\varepsilon\big)\Big)\Big)(x)\,dx\Big|\le$$

$$\Big(\int_\Omega\frac{1}{\varepsilon}|\varphi_\varepsilon(x)|^2\Big)^{1/2}\Big(\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\Big|\nabla\Big(div\Big(\Delta^{-1}\big(\varepsilon^2\Delta\zeta_\varepsilon-\varphi_\varepsilon\big)\Big)\Big)(x)\Big|^2dx\Big)^{1/2}\le$$

$$\Big(\int_\Omega\frac{1}{\varepsilon}|\varphi_\varepsilon(x)|^2\Big)^{1/2}\Big(\frac{2}{\varepsilon}\int_{\mathbb{R}^2}\Big|\nabla^2\Big(\Delta^{-1}\big(\varepsilon^2\Delta\zeta_\varepsilon-\varphi_\varepsilon\big)\Big)(x)\Big|^2dx\Big)^{1/2}=$$

$$\Big(\int_\Omega\frac{1}{\varepsilon}|\varphi_\varepsilon(x)|^2\Big)^{1/2}\Big(\int_\Omega\frac{2}{\varepsilon}\Big|\varepsilon^2\Delta\zeta_\varepsilon(x)-\varphi_\varepsilon(x)\Big|^2dx\Big)^{1/2}.$$

Then, since

$$(3.11)\quad \int_\Omega\frac{1}{\varepsilon}|\varphi_\varepsilon(x)|^2\le C\int_\Omega\frac{1}{\varepsilon}\Big|\int_{B_R(0)}l(z,x)\big(\bar{\varphi}(x+\varepsilon z)-\varphi(x)\big)dz\Big|\,dx\le$$

$$C\int_{B_R(0)}\frac{1}{\varepsilon}|l(z,x)|\Big(\int_\Omega\big|\bar{\varphi}(x+\varepsilon z)-\varphi(x)\big|dx\Big)\,dz$$

$$\le\bar{C}\|D\bar{\varphi}\|(\mathbb{R}^2)\int_{B_R(0)}|l(z,x)|\cdot|z|\,dz=O(1)\,,$$

using (3.9), from (3.10) we infer

$$(3.12)\quad \int_{\mathbb{R}^2} \frac{1}{\varepsilon}\Big|\nabla\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx \;=\; o_\varepsilon(1) + \int_\Omega \frac{1}{\varepsilon}\,\varphi_\varepsilon(x)\cdot\nabla\Big(div\big(\varepsilon^2\zeta_\varepsilon(x)\big)\Big)\,dx\,.$$

Next we remind that ξ_ε is defined by (3.3). By (3.7), we have,

$$(3.13)\qquad \Big|\nabla\Big(div\big(\varepsilon^2\zeta_\varepsilon(x)\big)\Big) - \xi_\varepsilon(x)\Big| \le \bar{C}\varepsilon^{2/3} \quad \forall x\in\Omega\,.$$

Then as before,

$$\Big|\int_\Omega \frac{1}{\varepsilon}\,\varphi_\varepsilon(x)\cdot\Big(\nabla\Big(div\big(\varepsilon^2\zeta_\varepsilon(x)\big)\Big) - \xi_\varepsilon(x)\Big)\,dx\Big| \le \\ \Big(\int_\Omega \frac{1}{\varepsilon}|\varphi_\varepsilon(x)|^2\Big)^{1/2}\Big(\int_\Omega \frac{1}{\varepsilon}\Big|\nabla\Big(div\big(\varepsilon^2\zeta_\varepsilon(x)\big)\Big) - \xi_\varepsilon(x)\Big|^2 dx\Big)^{1/2} \le C\varepsilon^{1/6}\,.$$

Therefore, from (3.12) we infer (3.4). □

Lemma 3.2. *Let $\varphi\in BV(\Omega,\mathbb{R})\cap L^\infty$ and $l\in\mathcal{U}$ (see Definition 2.3). For every $\varepsilon>0$ and every $x\in\mathbb{R}^2$ consider the function $\varphi_\varepsilon\in C^1(\mathbb{R}^2,\mathbb{R}^2)$ as in* (3.2). *Then,*

$$(3.14)\quad \lim_{\varepsilon\to 0}\int_{\mathbb{R}^2}\frac{1}{\varepsilon}\Big|\nabla\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx = \\ \int_{J_\varphi}\Big\{\int_{-\infty}^{+\infty}|\varphi^+(x)-\varphi^-(x)|^2\cdot\Big|\boldsymbol{\nu}(x)\cdot\int_t^{+\infty}q(s,x)ds\Big|^2 dt\Big\}d\mathcal{H}^1(x)\,,$$

where

$$(3.15)\qquad q(t,x) = \int_{H^0_{\boldsymbol{\nu}(x)}} l(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\,,$$

and $\boldsymbol{\nu}(x)$ is the jump vector of φ.

Proof. Step 1: We prove a useful expression,

$$(3.16)\quad \int_\Omega\frac{1}{\varepsilon}\varphi_\varepsilon(x)\cdot\xi_\varepsilon(x)\,dx = \\ \int_0^1\Big\{\int\limits_{\mathbb{R}^2}\Big(\frac{1}{t^2\varepsilon^2}\int\limits_{\Omega\cap B_{Rt\varepsilon}(y)}\Big\{\xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon},x\Big)\Big\}\,\frac{y-x}{t\varepsilon}\,dx\Big)\cdot d\,[D\bar{\varphi}](y)\Big\}dt\,,$$

where ξ_ε be as in (3.3). As before, we shall denote by $\nabla_1 l$ and $\nabla_2 l$ the gradient of l w.r.t. the first and second variables respectively. Denote $\big(\varphi_{t\varepsilon,1}(x),\varphi_{t\varepsilon,2}(x)\big) := \varphi_{t\varepsilon}(x)$

and $\big(l_1(z,x),l_2(z,x)\big) := l(z,x)$. Then for every $t\in(0,1]$, every $j\in\{1,2\}$ and every $x\in\mathbb{R}^2$ we have

$$\begin{aligned}(3.17)\quad \frac{d\big(\varphi_{t\varepsilon,j}(x)\big)}{dt} &= \frac{d}{dt}\bigg(\frac{1}{t^2\varepsilon^2}\int_{\mathbb{R}^2} l_j\Big(\frac{y-x}{t\varepsilon},x\Big)\bar\varphi(y)dy\bigg) = \\ &-\frac{1}{t^3\varepsilon^2}\int_{\mathbb{R}^2}\bigg\{\nabla_1 l_j\Big(\frac{y-x}{t\varepsilon},x\Big)\cdot\frac{y-x}{t\varepsilon}+2l_j\Big(\frac{y-x}{t\varepsilon},x\Big)\bigg\}\bar\varphi(y)dy \\ &= -\frac{1}{t^2\varepsilon}\int_{\mathbb{R}^2}\operatorname{div}_y\bigg\{l_j\Big(\frac{y-x}{t\varepsilon},x\Big)\frac{y-x}{t\varepsilon}\bigg\}\bar\varphi(y)dy \\ &= \frac{1}{t^2\varepsilon}\int_{\mathbb{R}^2} l_j\Big(\frac{y-x}{t\varepsilon},x\Big)\frac{y-x}{t\varepsilon}\cdot d\,[D\bar\varphi](y)\,.\end{aligned}$$

Therefore, for any $\rho\in(0,1)$ we have,

$$\begin{aligned}(3.18)\quad &\int_\Omega\frac{1}{\varepsilon}\big(\varphi_\varepsilon(x)-\varphi_{\rho\varepsilon}(x)\big)\cdot\xi_\varepsilon(x)\,dx = \int_\Omega\frac{1}{\varepsilon}\xi_\varepsilon(x)\cdot\bigg(\int_\rho^1\frac{d\big(\varphi_{t\varepsilon}(x)\big)}{dt}\bigg)\,dx = \\ &\int_\Omega\bigg\{\int_\rho^1\xi_\varepsilon(x)\cdot\bigg(\frac{1}{t^2\varepsilon^2}\int_{\mathbb{R}^2} l\Big(\frac{y-x}{t\varepsilon},x\Big)\Big\{\frac{y-x}{t\varepsilon}\cdot d\,[D\bar\varphi](y)\Big\}\bigg)dt\bigg\}dx = \\ &\int_\rho^1\bigg\{\int_\Omega\xi_\varepsilon(x)\cdot\bigg(\frac{1}{t^2\varepsilon^2}\int_{\mathbb{R}^2} l\Big(\frac{y-x}{t\varepsilon},x\Big)\Big\{\frac{y-x}{t\varepsilon}\cdot d\,[D\bar\varphi](y)\Big\}\bigg)dx\bigg\}dt = \\ &\quad\int_\rho^1\bigg\{\int\limits_{\mathbb{R}^2}\bigg(\frac{1}{t^2\varepsilon^2}\int\limits_{\Omega\cap B_{Rt\varepsilon}(y)}\Big\{\xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon},x\Big)\Big\}\,\frac{y-x}{t\varepsilon}\,dx\bigg)\cdot d\,[D\bar\varphi](y)\bigg\}dt\,.\end{aligned}$$

From our assumptions on φ, by (3.1), it follows that there exists a constant $C>0$, independent of ρ, such that $\big|\xi_\rho(x)\big|\le C$ for every $\rho>0$ and every $x\in\Omega$. Therefore, letting ρ tend to zero in (3.18), using the fact that $\lim_{\rho\to0}\|\varphi_\rho(x)\|_{L^1(\Omega)}=0$ (see (3.11)), we get (3.16).

Step 2: We prove the identity

$$\begin{aligned}(3.19)\quad &\int_{\mathbb{R}^2}\frac{1}{\varepsilon}\Big|\nabla\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2dx = \\ &o_\varepsilon(1)+\int\limits_0^1\bigg(\int\limits_{J_\varphi}\Big\{\int\limits_{B_R(0)}\big\{l(z,x)\cdot\xi_\varepsilon(x-\varepsilon tz)\big\}\,z\,dz\Big\}\cdot d\big[D\varphi\big](x)\bigg)dt \\ &\qquad+\int\limits_0^1\bigg(\int\limits_{G_\varphi}\Big\{\int\limits_{B_R(0)}\big\{l(z,x)\cdot\xi_\varepsilon(x-\varepsilon tz)\big\}\,z\,dz\Big\}\cdot d\big[D\varphi\big](x)\bigg)dt\,,\end{aligned}$$

where G_φ is the set of approximate continuity of φ. By Lemma 3.1 we have

$$(3.20)\qquad \int_{\mathbb{R}^2}\frac{1}{\varepsilon}\Big|\nabla\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2dx = o_\varepsilon(1)+\int_\Omega\frac{1}{\varepsilon}\,\varphi_\varepsilon(x)\cdot\xi_\varepsilon(x)\,dx\,.$$

So by (3.20) and (3.16),

$$(3.21)\quad \int_{\mathbb{R}^2} \frac{1}{\varepsilon}\Big|\nabla\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx = o_\varepsilon(1)+$$

$$\int_0^1 \left\{ \int\limits_{\mathbb{R}^2} \left(\frac{1}{t^2\varepsilon^2} \int\limits_{\Omega\cap B_{Rt\varepsilon}(y)} \Big\{\xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon},x\Big)\Big\} \frac{y-x}{t\varepsilon}\,dx\right)\cdot d\,[D\bar\varphi](y)\right\}dt = o_\varepsilon(1)+$$

$$\int_0^1 \left\{ \int\limits_{\mathbb{R}^2} \left(\frac{1}{t^2\varepsilon^2} \int\limits_{K\cap B_{Rt\varepsilon}(y)} \Big\{\xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon},x\Big)\Big\} \frac{y-x}{t\varepsilon}\,dx\right)\cdot d\,[D\bar\varphi](y)\right\}dt\,,$$

where $K \subset\subset \Omega$ is a compact set (see Definition 2.3). But, for every $\varepsilon < \frac{1}{R}\,dist(K,\partial\Omega)$ we have

$$\int\limits_{\mathbb{R}^2} \left(\frac{1}{t^2\varepsilon^2} \int\limits_{K\cap B_{Rt\varepsilon}(y)} \Big\{\xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon},x\Big)\Big\} \frac{y-x}{t\varepsilon}\,dx\right)\cdot d\,[D\bar\varphi](y) =$$

$$\int\limits_{\Omega} \left(\frac{1}{t^2\varepsilon^2} \int\limits_{K\cap B_{Rt\varepsilon}(y)} \Big\{\xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon},x\Big)\Big\} \frac{y-x}{t\varepsilon}\,dx\right)\cdot d\,[D\bar\varphi](y) =$$

$$\int\limits_{\Omega} \left(\frac{1}{t^2\varepsilon^2} \int\limits_{\mathbb{R}^2} \Big\{\xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon},x\Big)\Big\} \frac{y-x}{t\varepsilon}\,dx\right)\cdot d\,[D\bar\varphi](y)\,.$$

Therefore, by (3.21), we obtain

$$(3.22)\quad \int_{\mathbb{R}^2} \frac{1}{\varepsilon}\Big|\nabla\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx = o_\varepsilon(1)+$$

$$\int\limits_0^1 \int\limits_{\Omega} \left(\frac{1}{t^2\varepsilon^2} \int\limits_{\mathbb{R}^2} \Big\{\xi_\varepsilon(x)\cdot l\Big(\frac{y-x}{t\varepsilon},x\Big)\Big\} \frac{y-x}{t\varepsilon}\,dx\right)\cdot d\,[D\bar\varphi](y)\,dt = o_\varepsilon(1)+$$

$$\int\limits_0^1 \left(\int\limits_{\Omega} \Big\{ \int\limits_{B_R(0)} \big\{l(z,y-\varepsilon tz)\cdot\xi_\varepsilon(y-\varepsilon tz)\big\}\,z\,dz\Big\}\cdot d\big[D\varphi\big](y)\right)dt =$$

$$o_\varepsilon(1) + \int\limits_0^1 \left(\int\limits_{\Omega} \Big\{ \int\limits_{B_R(0)} \big\{l(z,x)\cdot\xi_\varepsilon(x-\varepsilon tz)\big\}\,z\,dz\Big\}\cdot d\big[D\varphi\big](x)\right)dt\,,$$

where in the last equality we used the estimate $|l(z,x-\varepsilon tz)-l(z,x)| \le C\varepsilon t|z|$. Therefore we obtain (3.19).

Step 3: We will prove that the second integral in the r.h.s of (3.19) vanishes as $\varepsilon \to 0$. For every x in G_φ we have,

$$\lim_{\rho\to 0^+} \frac{1}{\rho^2}\int_{B_\rho(x)} |\bar\varphi(y)-\tilde{\bar\varphi}(x)|\,dy = 0\,.$$

Taking $\rho = L\varepsilon$, for every $L > 0$, gives

$$\lim_{\varepsilon\to 0^+}\int_{B_L(0)} |\bar{\varphi}(x+\varepsilon z) - \tilde{\bar{\varphi}}(x)|\,dz = 0, \quad \text{for } x \text{ in } G_\varphi\,. \tag{3.23}$$

Using (3.1), since $\int_{\mathbb{R}^2}\nabla_z\big(div_z\, r(z, x-\varepsilon ty)\big)dz = 0$, for every x in G_φ, $y \in B_R(0)$, $t \in [0,1]$ and $L > 0$ we have,

$$\begin{aligned}
|\xi_\varepsilon(x-\varepsilon ty)| &= \left|\int_{\mathbb{R}^2}\nabla_z\big(div_z\, r(z,x-\varepsilon ty)\big)\bar{\varphi}(x+\varepsilon z-\varepsilon ty)\,dz\right| = \\
&\left|\int_{\mathbb{R}^2}\nabla_z\big(div_z\, r(z,x-\varepsilon ty)\big)\big(\bar{\varphi}(x+\varepsilon z-\varepsilon ty)-\tilde{\bar{\varphi}}(x)\big)\,dz\right| \leq \\
&\int_{B_L(0)}\Big|\nabla_z\big(div_z\, r(z,x-\varepsilon ty)\big)\Big|\cdot\big|\bar{\varphi}(x+\varepsilon z-\varepsilon ty)-\tilde{\bar{\varphi}}(x)\big|\,dz + \\
&\int_{\mathbb{R}^2\setminus B_L(0)}\Big|\nabla_z\big(div_z\, r(z,x-\varepsilon ty)\big)\Big|\cdot\big|\bar{\varphi}(x+\varepsilon z-\varepsilon ty)-\tilde{\bar{\varphi}}(x)\big|\,dz \\
&\leq A_L\int_{B_L(0)}|\bar{\varphi}(x+\varepsilon z-\varepsilon ty)-\tilde{\bar{\varphi}}(x)|\,dz + B\int_{\mathbb{R}^2\setminus B_L(0)}\frac{1}{|z|^3+1}dz \\
&\leq A_L\int_{B_{(L+R)}(0)}|\bar{\varphi}(x+\varepsilon z)-\tilde{\bar{\varphi}}(x)|\,dz + B\int_{\mathbb{R}^2\setminus B_L(0)}\frac{1}{|z|^3+1}dz\,,
\end{aligned} \tag{3.24}$$

where $B > 0$ constant and $A_L > 0$ depends only on L. Given $\delta > 0$ we can take $L > 0$ such that

$$B\int_{\mathbb{R}^2\setminus B_L(0)}\frac{1}{|z|^3+1}dz < \delta\,,$$

Then, using (3.24) and (3.23), we infer $\overline{\lim}_{\varepsilon\to 0^+}|\xi_\varepsilon(x-\varepsilon ty)| < \delta$ and since δ was arbitrary,

$$\lim_{\varepsilon\to 0^+}\xi_\varepsilon(x-\varepsilon ty) = 0 \quad \forall x\in G_\varphi,\ y\in B_R(0),\ t\in[0,1]. \tag{3.25}$$

Using (3.1), we also have $|\xi_\varepsilon(x-\varepsilon ty)| \leq C$, and therefore, plugging (3.25) into (3.19), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2}\frac{1}{\varepsilon}\Big|\nabla\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx = \\
&\qquad o_\varepsilon(1) + \int\limits_0^1\bigg(\int\limits_{J_\varphi}\Big\{\int\limits_{B_R(0)}\big\{l(z,x)\cdot\xi_\varepsilon(x-\varepsilon tz)\big\}\,z\,dz\Big\}\cdot d\big[D\varphi\big](x)\bigg)dt\,.
\end{aligned} \tag{3.26}$$

Step 4. Consider $\bar{l}(z,x) := \nabla_z\big(div_z\, r(z,x)\big)$. For every $\varepsilon, t \in (0,1)$, $x \in J_\varphi$ and $z \in B_R(0)$, we have

(3.27)

$$\xi_\varepsilon(x-\varepsilon tz) = \int_{\mathbb{R}^2} \bar{l}(y, x-\varepsilon tz)\bar{\varphi}\big(x+\varepsilon(y-tz)\big)\,dy = \int_{\mathbb{R}^2} \bar{l}(y+tz, x-\varepsilon tz)\bar{\varphi}\big(x+\varepsilon y\big)\,dy$$
$$= \int_{H_+(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)\bar{\varphi}\big(x+\varepsilon y\big)\,dy + \int_{H_-(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)\bar{\varphi}\big(x+\varepsilon y\big)\,dy$$
$$= \int_{H_+(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)\varphi^+(x)\,dy + \int_{H_-(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)\varphi^-(x)\,dy +$$
$$\int\limits_{H_+(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)\big(\bar{\varphi}(x+\varepsilon y)-\varphi^+(x)\big)\,dy + \int\limits_{H_-(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)\big(\bar{\varphi}(x+\varepsilon y)-\varphi^-(x)\big)\,dy\,.$$

By the definition of J_φ, for every $L>0$ we obtain,

$$\text{(3.28)}\qquad \begin{array}{l} \lim\limits_{\varepsilon\to 0^+} \displaystyle\int_{B_L^+(0,\boldsymbol{\nu}(x))} |\varphi(x+\varepsilon z)-\varphi^+(x)|\,dz = 0, \\ \lim\limits_{\varepsilon\to 0^+} \displaystyle\int_{B_L^-(0,\boldsymbol{\nu}(x))} |\varphi(x+\varepsilon z)-\varphi^-(x)|\,dz = 0. \end{array} \qquad \text{for } x\in J_\varphi\,.$$

Then, by (3.1), for every $L>R$ we have,

$$\text{(3.29)}\quad \left|\int\limits_{H_+(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)\big(\bar{\varphi}(x+\varepsilon y)-\varphi^+(x)\big)\,dy\right| \le$$
$$\int\limits_{B_L^+(0,\boldsymbol{\nu}(x))} |\bar{l}(y+tz, x-\varepsilon tz)|\cdot\big|\bar{\varphi}(x+\varepsilon y)-\varphi^+(x)\big|\,dy +$$
$$\int\limits_{H_+(0,\boldsymbol{\nu}(x))\setminus B_L^+(0,\boldsymbol{\nu}(x))} |\bar{l}(y+tz, x-\varepsilon tz)|\cdot\big|\bar{\varphi}(x+\varepsilon y)-\varphi^+(x)\big|\,dy$$
$$\le A_L \int\limits_{B_L^+(0,\boldsymbol{\nu}(x))} \big|\bar{\varphi}(x+\varepsilon y)-\varphi^+(x)\big|\,dy + B\int_{\mathbb{R}^2\setminus B_L(0)} \frac{1}{(|y|-R)^3+1}dy\,,$$

where $B>0$ constant and $A_L>0$ depends only on L. Given $\delta>0$ we can take $L>0$ such that

$$B\int_{\mathbb{R}^2\setminus B_L(0)} \frac{1}{(|y|-R)^3+1}dy < \delta\,,$$

Then, using (3.29) and (3.28), we infer

$$\overline{\lim_{\varepsilon\to 0^+}} \left|\int\limits_{H_+(0,\boldsymbol{\nu}(x))} \bar{l}(y+tz, x-\varepsilon tz)\big(\bar{\varphi}(x+\varepsilon y)-\varphi^+(x)\big)\,dy\right| < \delta\,,$$

and since δ was arbitrary,

(3.30)
$$\lim_{\varepsilon\to 0^+}\int\limits_{H_+(0,\boldsymbol{\nu}(x))}\bar{l}(y+tz,x-\varepsilon tz)\big(\bar{\varphi}(x+\varepsilon y)-\varphi^+(x)\big)\,dy=0\quad\forall x\in J_\varphi,\ z\in B_R(0),\ t\in[0,1].$$

By the same method,

(3.31)
$$\lim_{\varepsilon\to 0^+}\int\limits_{H_-(0,\boldsymbol{\nu}(x))}\bar{l}(y+tz,x-\varepsilon tz)\big(\bar{\varphi}(x+\varepsilon y)-\varphi^-(x)\big)\,dy=0\quad\forall x\in J_\varphi,\ z\in B_R(0),\ t\in[0,1].$$

Therefore, by (3.27) for every $\varepsilon,t\in(0,1)$, $x\in J_\varphi$ and $z\in B_R(0)$, we have

(3.32)
$$\begin{aligned}\xi_\varepsilon(x-\varepsilon tz)&=o_\varepsilon(1)+\varphi^+(x)\int_{H_+(0,\boldsymbol{\nu}(x))}\bar{l}(y+tz,x-\varepsilon tz)\,dy+\varphi^-(x)\int_{H_-(0,\boldsymbol{\nu}(x))}\bar{l}(y+tz,x-\varepsilon tz)\,dy\\&=o_\varepsilon(1)+\big(\varphi^+(x)-\varphi^-(x)\big)\int_{H_+(0,\boldsymbol{\nu}(x))}\bar{l}(y+tz,x-\varepsilon tz)\,dy\,,\end{aligned}$$

where we used the equality $\int_{\mathbb{R}^2}\bar{l}(y+tz,x-\varepsilon tz)dy=0$. Using (3.1), gives

$$\lim_{\varepsilon\to 0^+}\int_{H_+(0,\boldsymbol{\nu}(x))}\bar{l}(y+tz,x-\varepsilon tz)\,dy=\int_{H_+(0,\boldsymbol{\nu}(x))}\bar{l}(y+tz,x)\,dy.$$

Therefore, by (3.32), for every $x\in J_\varphi$, every $t\in(0,1)$ and every $z\in B_R(0)$, we obtain,

$$\lim_{\varepsilon\to 0^+}\xi_\varepsilon(x-\varepsilon tz)=\big(\varphi^+(x)-\varphi^-(x)\big)\int_{H_+(0,\boldsymbol{\nu}(x))}\bar{l}(y+tz,x)\,dy\,.\tag{3.33}$$

Note that

(3.34)
$$\begin{aligned}\int_{H_+(0,\boldsymbol{\nu}(x))}\bar{l}(y+tz,x)\,dy&=\int_{H_+(tz,\boldsymbol{\nu}(x))}\bar{l}(y,x)\,dy\\&=\int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty}\Big(\int_{H^0_{\boldsymbol{\nu}(x)}}\bar{l}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\Big)dt=\int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty}\bar{q}(\tau,x)\,d\tau,\end{aligned}$$

where

$$\bar{q}(t,x)=\int_{H^0_{\boldsymbol{\nu}(x)}}\bar{l}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\,.\tag{3.35}$$

Combining (3.33) and (3.34), for every $x\in J_\varphi$, every $t\in(0,1)$ and every $z\in B_R(0)$ we obtain,

$$\lim_{\varepsilon\to 0+}\xi_\varepsilon(x-\varepsilon tz)=\big(\varphi^+(x)-\varphi^-(x)\big)\int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty}\bar{q}(\tau,x)\,d\tau.\tag{3.36}$$

Using (3.36) in (3.26), we obtain,

$$(3.37)\quad \int_{\mathbb{R}^2} \frac{1}{\varepsilon}\Big|\nabla\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\Big|^2 dx = o_\varepsilon(1)+$$

$$\int_0^1 \Bigg(\int_{J_\varphi} \big(\varphi^+(x)-\varphi^-(x)\big)\Bigg\{\int_{B_R(0)} \bigg(l(z,x)\cdot\int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty} \bar{q}(\tau,x)\,d\tau\bigg)\,z\,dz\Bigg\}\cdot d\big[D\varphi\big](x)\Bigg)dt = o_\varepsilon(1)$$

$$+\int_{J_\varphi} \big|\varphi^+(x)-\varphi^-(x)\big|^2\Bigg\{\int_{B_R(0)} \bigg(l(z,x)\cdot\int_0^1\int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty} \bar{q}(\tau,x)\,d\tau dt\bigg)\big(\nu(x)\cdot z\big)\,dz\Bigg\}d\mathcal{H}^1(x)\,.$$

<u>Step 5:</u> We prove

$$(3.38)\qquad \bar{q}(t,x) = \big(q(t,x)\cdot\boldsymbol{\nu}(x)\big)\boldsymbol{\nu}(x)\,.$$

Consider $\big(r_1(z,x), r_2(z,x)\big) := r(z,x)$. Then, by (3.1), for every $k=1,2$, we obtain,

(3.39)

$$\int_{H^0_{\boldsymbol{\nu}(x)}} \nabla^2_z r_k(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y) = \int_{H^0_{\boldsymbol{\nu}(x)}} \boldsymbol{\nu}(x)\otimes\boldsymbol{\nu}(x)\,\frac{\partial^2 r_k}{\partial\big(\boldsymbol{\nu}(x)\big)^2}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)+$$

$$\int_{H^0_{\boldsymbol{\nu}(x)}} \big(\boldsymbol{\nu}(x)\otimes\boldsymbol{\nu}^\perp(x)+\boldsymbol{\nu}^\perp(x)\otimes\boldsymbol{\nu}(x)\big)\,\frac{\partial^2 r_k}{\partial\big(\boldsymbol{\nu}^\perp(x)\big)\partial\big(\boldsymbol{\nu}(x)\big)}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)$$

$$+\int_{H^0_{\boldsymbol{\nu}(x)}} \boldsymbol{\nu}^\perp(x)\otimes\boldsymbol{\nu}^\perp(x)\,\frac{\partial^2 r_k}{\partial\big(\boldsymbol{\nu}^\perp(x)\big)^2}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y) =$$

$$\int_{H^0_{\boldsymbol{\nu}(x)}} \boldsymbol{\nu}(x)\otimes\boldsymbol{\nu}(x)\,\frac{\partial^2 r_k}{\partial\big(\boldsymbol{\nu}(x)\big)^2}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\,,$$

where $\boldsymbol{\nu}^\perp(x)$ is the vector orthogonal to $\boldsymbol{\nu}(x)$ in $\mathbb{R}^2$ and all derivatives are taken in the first argument-z of $r(z,x)$. In particular

$$q(t,x) = \int_{H^0_{\boldsymbol{\nu}(x)}} l(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y) = \int_{H^0_{\boldsymbol{\nu}(x)}} \Delta r(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)$$

$$= \int_{H^0_{\boldsymbol{\nu}(x)}} \frac{\partial^2 r}{\partial\big(\boldsymbol{\nu}(x)\big)^2}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\,,$$

and

$$\bar{q}(t,x) = \int_{H^0_{\boldsymbol{\nu}(x)}} \bar{l}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y) = \int_{H^0_{\boldsymbol{\nu}(x)}} \nabla_z(div_z\, r)(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)$$

$$= \Bigg(\boldsymbol{\nu}(x)\cdot\int_{H^0_{\boldsymbol{\nu}(x)}} \frac{\partial^2 r}{\partial\big(\boldsymbol{\nu}(x)\big)^2}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\Bigg)\boldsymbol{\nu}(x)\,.$$

So, we obtain (3.38).

Step 6: Completing the proof. Plugging (3.38) into (3.37) gives

$$(3.40)\quad \int_{\mathbb{R}^2} \frac{1}{\varepsilon}\left|\nabla\big(div(\Delta^{-1}\varphi_\varepsilon)\big)(x)\right|^2 dx = o_\varepsilon(1) \;+\; \int\limits_{J_\varphi} \left|\varphi^+(x)-\varphi^-(x)\right|^2 \cdot$$

$$\cdot\Bigg\{\int\limits_{B_R(0)} \bigg(\big(l(z,x)\cdot\boldsymbol{\nu}(x)\big)\int_0^1\int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\bigg)\big(\nu(x)\cdot z\big)\,dz\Bigg\}d\mathcal{H}^1(x)\,.$$

Next we have

$$(3.41)\quad \int\limits_{B_R(0)} \bigg(\big(l(z,x)\cdot\boldsymbol{\nu}(x)\big)\int_0^1\int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\bigg)\big(\nu(x)\cdot z\big)\,dz =$$

$$\int\limits_{\mathbb{R}^2} \bigg(\big(l(z,x)\cdot\boldsymbol{\nu}(x)\big)\int_0^1\int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\bigg)\big(\nu(x)\cdot z\big)\,dz =$$

$$\int_{-\infty}^{+\infty} s\bigg(\int_0^1\int_{ts}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\bigg)\bigg(\int_{H^0_{\boldsymbol{\nu}(x)}}\big(l(s\boldsymbol{\nu}(x)+y,x)\cdot\boldsymbol{\nu}(x)\big)\,d\mathcal{H}^1(y)\bigg)ds$$

$$= \int_{-\infty}^{+\infty} s\big(q(s,x)\cdot\boldsymbol{\nu}(x)\big)\bigg(\int_0^1\int_{ts}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\bigg)\,ds$$

$$= \int_{-\infty}^{+\infty}\big(q(s,x)\cdot\boldsymbol{\nu}(x)\big)\bigg(\int_0^s\int_{t}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\bigg)\,ds\,.$$

Using the fact that $\int_{\mathbb{R}} q(\tau,x)d\tau = 0$ and integrating by path, we obtain,

$$(3.42)\quad \int_{-\infty}^{+\infty}\big(q(s,x)\cdot\boldsymbol{\nu}(x)\big)\bigg(\int_0^s\int_{t}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\bigg)\,ds$$

$$= \int_{-\infty}^{+\infty}\bigg(\int_{s}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau\bigg)^2 ds\,.$$

Therefore, returning to (3.41) we infer

$$(3.43)\quad \int\limits_{B_R(0)} \bigg(\big(l(z,x)\cdot\boldsymbol{\nu}(x)\big)\int_0^1\int_{t\boldsymbol{\nu}(x)\cdot z}^{+\infty}\big(q(\tau,x)\cdot\boldsymbol{\nu}(x)\big)\,d\tau dt\bigg)\big(\nu(x)\cdot z\big)\,dz$$

$$= \int_{-\infty}^{+\infty}\bigg(\int_{t}^{+\infty}\big(q(s,x)\cdot\boldsymbol{\nu}(x)\big)\,ds\bigg)^2 dt\,.$$

Plugging (3.43) in (3.40) gives the desired result (3.14). □

4. Proof of the main result

As before, throughout this section we assume that Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary. Next consider $u \in BV(\Omega, S^1)$, satisfying $div\, u = 0$ in Ω and $u\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ ($\boldsymbol{n}$ is the unit normal to $\partial\Omega$). Let $\varphi \in BV(\Omega,\mathbb{R})\cap L^\infty(\Omega,\mathbb{R})$, satisfying $u = e^{i\varphi}$ a e. in Ω. By [2, Proposition 3.21] we may extend φ to a function

$\bar{\varphi} \in BV(\mathbb{R}^2,\mathbb{R}) \cap L^\infty(\mathbb{R}^2,\mathbb{R})$ satisfying $\bar{\varphi} = \varphi$ a.e. in Ω, $\operatorname{supp}\bar{\varphi}$ is compact and $\|D\bar{\varphi}\|(\partial\Omega) = 0$ (from the proof of Proposition 3.21 in [2] it follows that if φ is bounded then its extension is also bounded). We also denote by $\bar{u} := e^{i\bar{\varphi}}$. Then $\bar{u} \in BV(\Omega'',\mathbb{R}^2) \cap L^\infty(\Omega'',\mathbb{R}^2)$ for some $\Omega'' \supset\supset \Omega$, satisfying $\bar{u} = u$ a.e. in Ω and, by Volpert's chain rule, $\|D\bar{u}\|(\partial\Omega) = 0$. Consider $\eta \in \mathcal{V}$. For any $\varepsilon > 0$ define a function $\psi_\varepsilon(x) : \mathbb{R}^2 \to \mathbb{R}$ by

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \eta\Big(\frac{y-x}{\varepsilon}, x\Big)\bar{\varphi}(y)dy = \int_{\mathbb{R}^2} \eta(z,x)\bar{\varphi}(x+\varepsilon z)\,dz, \quad \forall x \in \mathbb{R}^2\,. \tag{4.1}$$

Proposition 4.1. *Let u, φ, $\bar{u}$, $\bar{\varphi}$ and η be as above. Consider $\psi_\varepsilon(x)$ defined by (4.1). Then,*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2} \Big|\nabla\, div\, \Delta^{-1}\big(\chi_\Omega(x)e^{i\psi_\varepsilon(x)}\big)\Big|^2 dx = \int_{J_\varphi}\left\{\int_{-\infty}^{+\infty} \Big|\boldsymbol{\nu}(x)\cdot\Big(e^{i\gamma(t,x)} - e^{i\varphi^-(x)}\Big)\Big|^2 dt\right\}d\mathcal{H}^1(x)\,, \tag{4.2}$$

where

$$\gamma(t,x) = \varphi^-(x)\int_{-\infty}^{t} p(s,x)ds + \varphi^+(x)\int_t^{+\infty} p(s,x)\,ds\,, \tag{4.3}$$

with

$$p(t,x) = \int_{H^0_{\boldsymbol{\nu}(x)}} \eta(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\,, \tag{4.4}$$

and χ_Ω is the indicator function of Ω.

Proof. Since $(u^+ - u^-)\cdot\boldsymbol{\nu} = 0$, the r.h.s. in (4.2) does not depend on the orientation of J_φ, we may assume that $\boldsymbol{\nu}(x)$ is Borel measurable.

Together with $\eta \in \mathcal{V}$ we consider a second kernel $\bar{\eta} \in \mathcal{V}$. Let

$$\bar{p}(t,x) = \int_{H^0_{\boldsymbol{\nu}(x)}} \bar{\eta}(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\,. \tag{4.5}$$

For any $\varepsilon > 0$ define a function $u_\varepsilon(x) : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$u_\varepsilon(x) := \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \bar{\eta}\Big(\frac{y-x}{\varepsilon}, x\Big)\bar{u}(y)\,dy = \int_{\mathbb{R}^2} \bar{\eta}(z,x)\,e^{i\bar{\varphi}(x+\varepsilon z)}\,dz, \quad \forall x \in \mathbb{R}^2\,. \tag{4.6}$$

Define $Q : \mathbb{R}\times J_\varphi \to \mathbb{R}^2$ by

$$Q(t,x) := e^{i\gamma(t,x)} - \left(\Big\{\int_{-\infty}^{t}\bar{p}(s,x)ds\Big\}e^{i\varphi^-(x)} + \Big\{\int_t^{+\infty}\bar{p}(s,x)\,ds\Big\}e^{i\varphi^+(x)}\right), \tag{4.7}$$

where $\gamma(t,x)$ is defined by (4.3). Then define $q:\mathbb{R}\times\Omega\to\mathbb{R}^2$ by

$$
q(t,x)=\begin{cases}-\frac{1}{(\varphi^+(x)-\varphi^-(x))}\frac{dQ(t,x)}{dt} & x\in J_\varphi\,,\\ 0 & x\in\Omega\setminus J_\varphi\,.\end{cases} \tag{4.8}
$$

Then $q(t,x)$ is Borel measurable, q is bounded on $\mathbb{R}\times\Omega$, there exists $M>0$ such that $\operatorname{supp} q\subset[-M,M]\times\Omega$ and $\int_{\mathbb{R}} q(t,x)\,dt=0\ \forall x\in\Omega$. Moreover

$$
\big(\varphi^+(x)-\varphi^-(x)\big)\int_t^{+\infty} q(s,x)ds=Q(t,x)\,. \tag{4.9}
$$

Then by Lemma 2.2, there exists a sequence of functions $l_n\in\mathcal{U}$ (see Definition 2.3), such that the sequence of functions $\{q_n\}$ defined on $\mathbb{R}\times\Omega$ by

$$
q_n(t,x)=\int_{H^0_{\boldsymbol{\nu}_0(x)}} l_n(t\boldsymbol{\nu}_0(x)+y,x)d\mathcal{H}^1(y),
$$

has the following properties:

$$
\text{there exists } C_0 \text{ such that } \|q_n\|_{L^\infty}\le C_0, \tag{4.10}
$$

$$
\text{there exists } M>0 \text{ such that } q_n(t,x)=0 \text{ for } |t|>M, \text{ and every } x\in\Omega, \tag{4.11}
$$

$$
\lim_{n\to\infty}\int_\Omega\int_{\mathbb{R}}|q_n(t,x)-q(t,x)|\,dt\,d\|D\varphi\|(x)=0. \tag{4.12}
$$

In particular,

$$
\lim_{n\to\infty}\int_{J_\varphi}\int_{\mathbb{R}}\big|\varphi^+(x)-\varphi^-(x)\big|\cdot|q_n(t,x)-q(t,x)|\,dt\,d\mathcal{H}^1(x)=0\,. \tag{4.13}
$$

For every positive integer n and for every $\varepsilon>0$ consider the function $\varphi_{n,\varepsilon}\in C^1(\mathbb{R}^2,\mathbb{R}^2)$ given by

$$
\varphi_{n,\varepsilon}(x):=\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} l_n\Big(\frac{y-x}{\varepsilon},x\Big)\bar\varphi(y)\,dy=\int_{\mathbb{R}^2} l_n(z,x)\bar\varphi(x+\varepsilon z)\,dz\,, \tag{4.14}
$$

Next, we will use the following inequality, valid for any $f(x),g(x),\lambda(x)\in L^2(\mathbb{R}^2,\mathbb{R}^2)$,

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^2}|f(x)|^2dx-\int_{\mathbb{R}^2}|g(x)|^2dx\Big| &\le\\
&\Big(\|f-g-\lambda\|_{L^2}+\|\lambda\|_{L^2}\Big)\sqrt{2\Big(\int_{\mathbb{R}^2}|f(x)|^2dx+\int_{\mathbb{R}^2}|g(x)|^2dx\Big)}\,.
\end{aligned} \tag{4.15}
$$

Therefore, since $\varphi_{n,\varepsilon}(x)=0$ for $x\notin\Omega$ and since $div\big(\chi_\Omega\bar u\big)=0$ as a distribution, we obtain,

$$(4.16)\quad \left|\int_{\mathbb{R}^2}\Big|\nabla\, div\,\Delta^{-1}\big(\chi_\Omega(x)e^{i\psi_\varepsilon(x)}\big)\Big|^2dx-\int_{\mathbb{R}^2}\Big|\nabla\, div\,\Delta^{-1}\big(\varphi_{n,\varepsilon}(x)\big)\Big|^2dx\right|\leq$$

$$2\bigg(\Big\|\nabla\, div\,\Delta^{-1}\Big(\chi_\Omega\big(e^{i\psi_\varepsilon}-\varphi_{n,\varepsilon}-u_\varepsilon\big)\Big)\Big\|_{L^2}+\big\|\nabla\, div\,\Delta^{-1}(\chi_\Omega u_\varepsilon)\big\|_{L^2}\bigg)\cdot$$

$$\cdot\sqrt{\int_{\mathbb{R}^2}\Big|\nabla\, div\,\Delta^{-1}\big(\chi_\Omega e^{i\psi_\varepsilon}\big)\Big|^2dx+\int_{\mathbb{R}^2}\Big|\nabla\, div\,\Delta^{-1}\big(\varphi_{n,\varepsilon}\big)\Big|^2dx}=$$

$$2\bigg(\Big\|\nabla\, div\,\Delta^{-1}\Big(\chi_\Omega\big(e^{i\psi_\varepsilon}-\varphi_{n,\varepsilon}-u_\varepsilon\big)\Big)\Big\|_{L^2}+\Big\|\nabla\, div\,\Delta^{-1}\big(\chi_\Omega(u_\varepsilon-\bar u)\big)\Big\|_{L^2}\bigg)\cdot$$

$$\cdot\sqrt{\int_{\mathbb{R}^2}\Big|\nabla\, div\,\Delta^{-1}\big(\chi_\Omega(e^{i\psi_\varepsilon}-e^{i\bar\varphi})\big)\Big|^2dx+\int_{\mathbb{R}^2}\Big|\nabla\, div\,\Delta^{-1}\big(\varphi_{n,\varepsilon}\big)\Big|^2dx}\,.$$

But since for every $f\in L^\infty(\mathbb{R}^2,\mathbb{R}^2)$ with compact support we have

$$\int_{\mathbb{R}^2}\big|\nabla\, div\,\Delta^{-1}f\big|^2dx\leq 2\int_{\mathbb{R}^2}\big|\nabla^2\,\Delta^{-1}f\big|^2dx=2\int_{\mathbb{R}^2}\big|f\big|^2dx\,,$$

by (4.16), we obtain

$$(4.17)\quad \left|\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\Big|\nabla\, div\,\Delta^{-1}\big(\chi_\Omega(x)e^{i\psi_\varepsilon(x)}\big)\Big|^2dx-\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\Big|\nabla\, div\,\Delta^{-1}\big(\varphi_{n,\varepsilon}(x)\big)\Big|^2dx\right|\leq$$

$$4\left(\sqrt{\frac{1}{\varepsilon}\int_\Omega\big|e^{i\psi_\varepsilon}-\varphi_{n,\varepsilon}-u_\varepsilon\big|^2dx}+\sqrt{\frac{1}{\varepsilon}\int_\Omega|u_\varepsilon-u|^2dx}\right)\cdot$$

$$\cdot\sqrt{\frac{1}{\varepsilon}\int_\Omega\big|e^{i\psi_\varepsilon}-e^{i\varphi}\big|^2dx+\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\Big|\nabla\, div\,\Delta^{-1}\big(\varphi_{n,\varepsilon}\big)\Big|^2dx}\,.$$

Therefore, setting

$$L_0:=\int_{J_\varphi}\left\{\int_{-\infty}^{+\infty}\Big|\boldsymbol{\nu}(x)\cdot\Big(e^{i\gamma(t,x)}-e^{i\varphi^-(x)}\Big)\Big|^2dt\right\}d\mathcal{H}^1(x)\,,$$

we have

$$(4.18)\quad \left|\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\Big|\nabla\, div\,\Delta^{-1}\big(\chi_\Omega e^{i\psi_\varepsilon}\big)\Big|^2dx-L_0\right|\leq\left|L_0-\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\big|\nabla\, div\,\Delta^{-1}(\varphi_{n,\varepsilon})\big|^2dx\right|$$

$$+4\left(\sqrt{\frac{1}{\varepsilon}\int_\Omega\big|e^{i\psi_\varepsilon}-\varphi_{n,\varepsilon}-u_\varepsilon\big|^2dx}+\sqrt{\frac{1}{\varepsilon}\int_\Omega|u_\varepsilon-u|^2dx}\right)\cdot$$

$$\cdot\sqrt{\frac{1}{\varepsilon}\int_\Omega\big|e^{i\psi_\varepsilon}-e^{i\varphi}\big|^2dx+\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\Big|\nabla\, div\,\Delta^{-1}\big(\varphi_{n,\varepsilon}\big)\Big|^2dx}\,.$$

By Proposition 2.1, we obtain,

$$\text{(4.19)}\quad \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega \left|e^{i\psi_\varepsilon}-\varphi_{n,\varepsilon}-u_\varepsilon\right|^2dx = D_n :=$$

$$\int_{J_\varphi}\left\{\int_{-\infty}^{+\infty}\left|e^{i\gamma(t,x)}-\big(\varphi^+(x)-\varphi^-(x)\big)\int_t^{+\infty}q_n(s,x)ds-\Gamma(t,x)\right|^2dt\right\}d\mathcal{H}^1(x)\,,$$

where $\gamma(t,x)$ is defined by (4.3), and

$$\Gamma(t,x):=\Big\{\int_{-\infty}^t\bar{p}(s,x)ds\Big\}e^{i\varphi^-(x)}+\Big\{\int_t^{+\infty}\bar{p}(s,x)\,ds\Big\}e^{i\varphi^+(x)}\,.$$

By Proposition 2.1, we also infer,

$$\text{(4.20)}\quad \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega |u_\varepsilon-u|^2dx = T(\bar{\eta}) :=$$

$$\int_{J_\varphi}\left\{\int_{-\infty}^{0}\left|\Gamma(t,x)-u^+(x)\right|^2dt+\int_0^{+\infty}\left|\Gamma(t,x)-u^-(x)\right|^2dt\right\}d\mathcal{H}^1(x)=$$

$$\int_{J_\varphi}\left\{\int_{-\infty}^{0}\left|(u^+-u^-)\int_{-\infty}^t\bar{p}(s,\cdot)ds\right|^2dt+\int_0^{+\infty}\left|(u^+-u^-)\int_t^{+\infty}\bar{p}(s,\cdot)\,ds\right|^2dt\right\}d\mathcal{H}^1\,,$$

and

$$\text{(4.21)}\quad \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega \left|e^{i\psi_\varepsilon}-e^{i\varphi}\right|^2dx = M :=$$

$$\int_{J_\varphi}\left\{\int_{-\infty}^{0}\left|e^{i\gamma(t,x)}-e^{i\varphi^+(x)}\right|^2dt+\int_0^{+\infty}\left|e^{i\gamma(t,x)}-e^{i\varphi^-(x)}\right|^2dt\right\}d\mathcal{H}^1(x)\,.$$

By Lemma 3.2 we obtain

$$\text{(4.22)}\quad \lim_{\varepsilon\to 0}\int_{\mathbb{R}^2}\frac{1}{\varepsilon}\left|\nabla\big(div(\Delta^{-1}\varphi_{n,\varepsilon})\big)(x)\right|^2dx = L_n :=$$

$$\int_{J_\varphi}\left\{\int_{-\infty}^{+\infty}|\varphi^+(x)-\varphi^-(x)|^2\cdot\left|\boldsymbol{\nu}(x)\cdot\int_t^{+\infty}q_n(s,x)ds\right|^2dt\right\}d\mathcal{H}^1(x)\,.$$

Therefore, letting ε tend to 0 in (4.18), we obtain,

$$\text{(4.23)}\quad \overline{\lim_{\varepsilon\to 0^+}}\left|\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\left|\nabla\, div\,\Delta^{-1}\big(\chi_\Omega e^{i\psi_\varepsilon}\big)\right|^2dx-L_0\right|$$

$$\leq |L_0-L_n|+4\Big(\sqrt{D_n}+\sqrt{T(\bar{\eta})}\Big)\sqrt{M+L_n}\,.$$

Using (4.7), (4.9), (4.13), (4.10) and (4.11) we obtain

$$\text{(4.24)}\qquad \lim_{n\to\infty}D_n=0\,,$$

and since $\big(u^+(x)-u^-(x)\big)\perp\boldsymbol{\nu}(x)$ (by $div\,u=0$), we also infer

$$\text{(4.25)}\qquad \lim_{n\to\infty}L_n=L_0:=\int_{J_\varphi}\left\{\int_{-\infty}^{+\infty}\left|\boldsymbol{\nu}(x)\cdot\Big(e^{i\gamma(t,x)}-e^{i\varphi^-(x)}\Big)\right|^2dt\right\}d\mathcal{H}^1(x)\,.$$

Therefore, letting n tend to $+\infty$ in (4.23), we obtain,

(4.26)
$$\varlimsup_{\varepsilon\to 0^+}\left|\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\Big|\nabla\, div\,\Delta^{-1}\big(\chi_\Omega e^{i\psi_\varepsilon}\big)\Big|^2dx-\int_{J_\varphi}\left\{\int_{-\infty}^{+\infty}\Big|\boldsymbol{\nu}\cdot\Big(e^{i\gamma(t,\cdot)}-e^{i\varphi^-}\Big)\Big|^2dt\right\}d\mathcal{H}^1\right|$$
$$\le 4\sqrt{T(\bar\eta)}\sqrt{M+L_0}\,.$$

This equation is valid for any $\bar\eta\in\mathcal{V}$, and the constants M and L_0 do not depend on $\bar\eta$. For every $\delta>0$ we always can choose $\bar\eta_\delta\in C^2(\mathbb{R}^2\times\mathbb{R}^2,\mathbb{R})$, satisfying $\bar\eta_\delta\ge 0$, $supp\,\bar\eta_\delta\subset B_\delta(0)\times\Omega'$ and $\int_{\mathbb{R}^2}\bar\eta_\delta(z,x)dz=1$ for any $x\in\Omega$. Then, as before, define $\bar p_\delta(t,x):\mathbb{R}\times J_\varphi\to\mathbb{R}$ by

$$\bar p_\delta(t,x)=\int_{H^0_{\boldsymbol{\nu}(x)}}\bar\eta_\delta(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^1(y)\,.$$

Since $\bar p_\delta\ge 0$ and $\operatorname{supp}\bar p_\delta(t,x)\subset[-\delta,\delta]\times J_\varphi$ and $\int_{-\infty}^{\infty}\bar p_\delta(t,x)dt=1$, by (4.20) we infer

$$T(\bar\eta_\delta)\le$$
$$\int_{J_\varphi}\left\{\int_{-\delta}^{0}\left|(u^+-u^-)\int_{-\infty}^{t}\bar p_\delta(s,\cdot)ds\right|^2dt+\int_0^\delta\left|(u^+-u^-)\int_t^{+\infty}\bar p_\delta(s,\cdot)\,ds\right|^2dt\right\}d\mathcal{H}^1$$
$$\le 2\delta\int_{J_\varphi}|u^+-u^-|^2d\mathcal{H}^1\le 4\delta\int_{J_\varphi}|u^+-u^-|d\mathcal{H}^1\le 4\delta\|Du\|(\Omega)\,.$$

Therefore, by (4.26) we obtain

(4.27)
$$\varlimsup_{\varepsilon\to 0^+}\left|\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\Big|\nabla\, div\,\Delta^{-1}\big(\chi_\Omega e^{i\psi_\varepsilon}\big)\Big|^2dx-\int_{J_\varphi}\left\{\int_{-\infty}^{+\infty}\Big|\boldsymbol{\nu}\cdot\Big(e^{i\gamma(t,\cdot)}-e^{i\varphi^-}\Big)\Big|^2dt\right\}d\mathcal{H}^1\right|$$
$$\le 8\sqrt{\delta}\sqrt{\|Du\|(\Omega)}\sqrt{M+L_0}\,.$$

For $\delta\to 0$, (4.27) gives (4.2). $\square$

Let φ, $\bar\varphi$ and η be as in Proposition 4.1 and ψ_ε be defined by (4.1). Then using [16, Proposition 3.1], we obtain,

$$\lim_{\varepsilon\to 0}\int_\Omega\varepsilon|\nabla\psi_\varepsilon(x)|^2\,dx=\int_{J_\varphi}\Big|\varphi^+(x)-\varphi^-(x)\Big|^2\cdot\left(\int_{\mathbb{R}}p^2(t,x)dt\right)d\mathcal{H}^1(x)\,,\tag{4.28}$$

where $p(t,x)$ is defined by (4.4). As in [16] and [17] we also easily deduce that

$$\lim_{\varepsilon\to 0^+}\psi_\varepsilon(x)=\varphi(x)\quad\text{in}\quad L^p(\Omega,\mathbb{R})\quad\forall p\in[1,\infty)\,.$$

Combining these facts with the result of Proposition 4.1, we infer the following.

Corollary 4.1. *Let $u \in BV(\Omega, S^1)$, satisfying $div\, u = 0$ in Ω and $u \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ ($\boldsymbol{n}$ is the unit normal to $\partial\Omega$). Let $\varphi \in BV(\Omega, \mathbb{R}) \cap L^\infty$ such that $u = e^{i\varphi}$ a e. in Ω. Consider a function $\bar\varphi \in BV(\mathbb{R}^2, \mathbb{R}) \cap L^\infty$ such that $\bar\varphi = \varphi$ a.e. in Ω, $\operatorname{supp}\bar\varphi$ is compact and $\|D\bar\varphi\|(\partial\Omega) = 0$. Given $\eta \in \mathcal{V}$, for every $\varepsilon > 0$ let ψ_ε be defined by (4.1). Then,*

$$(4.29)\quad \lim_{\varepsilon\to 0}\left(\int_\Omega \varepsilon\left|\nabla e^{i\psi_\varepsilon(x)}\right|^2 dx + \int_{\mathbb{R}^2}\left|\nabla\, div\, \Delta^{-1}\big(\chi_\Omega(x)e^{i\psi_\varepsilon(x)}\big)\right|^2 dx\right) =$$

$$Y_\varphi(\eta) := \int_{J_\varphi}\left|\varphi^+(x) - \varphi^-(x)\right|^2 \cdot \left(\int_{-\infty}^{+\infty} p^2(t,x)dt\right) d\mathcal{H}^1(x)$$
$$+ \int_{J_\varphi}\left\{\int_{-\infty}^{+\infty}\left|\boldsymbol{\nu}(x)\cdot\left(e^{i\gamma(t,x)} - e^{i\varphi^-(x)}\right)\right|^2 dt\right\} d\mathcal{H}^1(x),$$

where γ and p defined by (4.3) and (4.4) respectively. Moreover,

$$\lim_{\varepsilon\to 0^+}\psi_\varepsilon(x) = \varphi(x) \quad in \quad L^p(\Omega,\mathbb{R}) \quad \forall p\in[1,\infty).$$

Next we turn to the minimization problem of the term on the r.h.s. of (4.29), over all kernels $\eta \in \mathcal{V}$, analogously to that was done in [16] and [17]. By the same method, as there, we can obtain the following.

Lemma 4.1. *Let $Y_\varphi(\eta) : \mathcal{V} \to \mathbb{R}$ be defined as the r.h.s. of (4.29). Then,*

$$(4.30)\quad \inf_{\eta\in\mathcal{V}} Y_\varphi(\eta) = \mathcal{J}_0(\varphi) :=$$
$$\int_{J_\varphi} 2\left|\varphi^+(x) - \varphi^-(x)\right|\left\{\int_0^1\left|\boldsymbol{\nu}(x)\cdot\left(e^{i\left(s\varphi^-(x)+(1-s)\varphi^+(x)\right)} - e^{i\varphi^-(x)}\right)\right|^2 ds\right\} d\mathcal{H}^1(x)$$
$$= \int_{J_\varphi} 2\left|\int_{\varphi^-(x)}^{\varphi^+(x)}\left|\boldsymbol{\nu}(x)\cdot\left(e^{it} - e^{i\varphi^-(x)}\right)\right|^2 dt\right| d\mathcal{H}^1(x).$$

By [19, (II.36)] we infer that

$$(4.31)\quad \mathcal{J}_0(\varphi) = \int_{J_\varphi} 2\left|\int_{\varphi^-(x)}^{\varphi^+(x)}\left|\boldsymbol{\nu}(x)\cdot\left(e^{it} - e^{i\varphi^-(x)}\right)\right|^2 dt\right| d\mathcal{H}^1(x)$$
$$= 2\int_{\mathbb{R}}\int_\Omega\left|div_x\, T^t u\right| dx\, dt,$$

where we (as in [19]), consider $T^t\varphi := \inf(\varphi, t)$ and $T^t u := e^{i T^t\varphi}$.

Proof of Theorem 1.2. The case of $\varphi \in BV(\Omega, \mathbb{R}) \cap L^\infty$ follows easily from Corollary 4.1 and Lemma 4.1 by using a standard diagonal argument as in the proofs of [17, Theorem 1.1 and Theorem 1.2].

It remains to consider the case of an unbounded $\varphi \in BV(\Omega,\mathbb{R})$, such that $e^{i\varphi(x)} = u(x)$ a.e. in Ω. First recall that by [6] there exists $\varphi_0 \in BV(\Omega,\mathbb{R})\cap L^\infty(\Omega,\mathbb{R})$ satisfying $e^{i\varphi_0(x)} = u(x)$ a.e. in Ω. Then $\varphi(x) = \varphi_0(x) + 2\pi l(x)$ where $l \in BV(\Omega,\mathbb{Z})$. For each integer $n \geq 1$ define,

$$l_n(x) := \begin{cases} l(x) & x \in \Omega,\ |l(x)| \leq n, \\ n & x \in \Omega,\ l(x) > n, \\ -n & x \in \Omega,\ l(x) < -n\,, \end{cases} \qquad \varphi_n(x) := \varphi_0(x) + 2\pi l_n(x)\,.$$

Clearly $\varphi_n \in BV(\Omega)\cap L^\infty(\Omega)$ and $e^{i\varphi_n(x)} = u(x)$ a.e. in Ω. From the case of a bounded φ, considered above, it follows that for each n there exists a family $\{v_{n,\varepsilon}\}_{\varepsilon>0} \subset C^2(\Omega,\mathbb{R})$ satisfying $\lim_{\varepsilon\to 0} v_{n,\varepsilon}(x) = \varphi_n(x)$ in $L^1(\Omega,\mathbb{R})$ and

$$\lim_{\varepsilon\to 0}\left(\int_\Omega \varepsilon\left|\nabla e^{iv_{n,\varepsilon}(x)}\right|^2 dx + \int_{\mathbb{R}^2}\left|\nabla\, div\, \Delta^{-1}\left(\chi_\Omega(x)e^{iv_{n,\varepsilon}(x)}\right)\right|^2 dx\right) =$$

$$\mathcal{J}_0(\varphi_n) = \int_{J_{\varphi_n}} 2\left|\int_{\varphi_n^-(x)}^{\varphi_n^+(x)} \left|\,\boldsymbol{\nu}_n(x)\cdot\left(e^{it} - e^{i\varphi_n^-(x)}\right)\right|^2 dt\right| d\mathcal{H}^1(x)\,.$$

Since for any $x \in \Omega$ we have $|\varphi_n(x)| \leq |\varphi_0(x)| + 2\pi|l(x)|$ while $\varphi_n(x) = \varphi(x)$ for n sufficiently large, we deduce by dominated convergence that

$$\lim_{n\to\infty}\varphi_n(x) = \varphi(x) \quad \text{in } L^1(\Omega,\mathbb{R})\,.$$

Put $\lambda_n(x) := |\varphi_n^+(x) - \varphi_n^-(x)|$. For $\mathcal{H}^{N-1}$-almost every $x \in J_{\varphi_0}\cup J_l$ we have $\lambda_n(x) \leq \left|\varphi_0^+(x) - \varphi_0^-(x)\right| + 2\pi\left|l^+(x) - l^-(x)\right|$, while $\lambda_n(x) = |\varphi^+(x) - \varphi^-(x)|$ for sufficiently large n. Moreover, $\mathcal{H}^{N-1}\left(J_{\varphi_n}\setminus(J_{\varphi_0}\cup J_l)\right) = 0$ and $\boldsymbol{\nu}_n(x) = \boldsymbol{\nu}(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in J_{\varphi_n}\cap J_\varphi$, for each n. Therefore, by dominated convergence,

$$\lim_{n\to\infty}\int_{J_{\varphi_n}} 2\left|\int_{\varphi_n^-(x)}^{\varphi_n^+(x)} \left|\,\boldsymbol{\nu}_n(x)\cdot\left(e^{it} - e^{i\varphi_n^-(x)}\right)\right|^2 dt\right| d\mathcal{H}^1(x)$$
$$= \int_{J_\varphi} 2\left|\int_{\varphi^-(x)}^{\varphi^+(x)} \left|\,\boldsymbol{\nu}(x)\cdot\left(e^{it} - e^{i\varphi^-(x)}\right)\right|^2 dt\right| d\mathcal{H}^1(x)\,.$$

To complete the proof, we apply to $\{v_{n,\varepsilon}\}$ a standard diagonal argument. □

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